MOMENTS OF AN EXPONENTIAL SUM RELATED TO THE DIVISOR FUNCTION

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Abstract. We use the circle method to obtain tight bounds on the $L^p$ norm of an exponential sum involving the divisor function for $p > 2$.

1. INTRODUCTION

Let $X \geq 1$ be sufficiently large. For a function $f : \mathbb{N} \to \mathbb{C}$, let

$$M_f(\alpha) = \sum_{n \leq X} f(n) e(n\alpha)$$

where as usual, $e(\alpha) := e^{2\pi i \alpha}$. Information on the structure of $f(n)$ can be obtained by studying the size of $L^p$-integrals of $M_f(\alpha)$, and bounds on them are often useful in applications of the circle method. Write

$$(1.1) \quad I_f(p) := \int_0^1 |M_f(\alpha)|^p d\alpha.$$

In the case $p = 1, f = \tau$, it was shown in [GP] that

$$(1.2) \quad \sqrt{X} \ll I_\tau(1) \ll \sqrt{X} \log X.$$ 

where

$$\tau(n) := \sum_{d \mid n} 1.$$

For sequences other than $\tau(n)$, similar results have been established in the case $p = 1$. For example, with $\mu$ the M"obius function, we have that $X^{1/6} \ll I_\mu(1) \ll X^{1/2}$ where the upper bound follows from Parseval’s identity, and the lower bound follows from Theorem 3 in [BR]. Estimates for $I_f(1)$ in the case $f$ is an indicator function for the primes have been obtained by Vaughan [Va1] and Goldston [Go], and in the case $f$ is the indicator function for integers not divisible by the $r$th power of any prime by Balog and Ruzsa [BR] (in fact, a result of Keil [Ke] finds the exact order of magnitude for all moments but $1 + \frac{1}{r}$ in which case the exact order of magnitude is found within a factor of $\log X$).

In this paper, we shall focus on the case $f = \tau$, the divisor function. Note that we have that by Parseval’s identity

$$(1.3) \quad I_\tau(2) = \sum_{n \leq X} \tau(n)^2 \sim \frac{1}{\pi^2} X (\log X)^3.$$ 

In this paper, we shall obtain tight estimates on $I_\tau(p)$ for $p > 2$. In particular, we prove the following result.
Theorem 1.1. We have that for \( p > 2 \)

\[
\int_0^1 |M_\tau(\alpha)|^p d\alpha \asymp_p X^{p-1} (\log X)^p.
\]

Throughout this paper, all implied constants will be assumed to depend only on \( p \) unless otherwise specified.

2. Preliminaries and setup

Note that we have that

\[
M_\tau(\alpha) = \sum_{n \leq X} \tau(n)e(n\alpha) = 2 \sum_{uv \leq X, \quad u < v} e(\alpha uv) + \sum_{uv \leq X, \quad u = v} e(\alpha u^2).
\]

Also, let

\[
v(\beta) := \sum_{n \leq X} e(n\beta).
\]

We record the following well-known bound on \( v(\beta) \) which we will use later.

Lemma 2.1. We have that for \( \beta \notin \mathbb{Z} \), \( v(\beta) \asymp \min(X, \|\beta\|^{-1}) \) where for \( \alpha \in \mathbb{R} \), we let \( \|\alpha\| := \inf_{n \in \mathbb{Z}} |\alpha - n| \).

In addition, we shall also use the following result on moments of \( v(\beta) \).

Lemma 2.2. For \( p > 2 \), we have that

\[
\int_0^1 |v(\beta)|^p d\beta \asymp X^{p-1}.
\]

Proof. Note that by Lemma 2.1, we have that

\[
\int_0^1 |v(\beta)|^p d\beta \geq \int_{-X^{-1}}^{X^{-1}} |v(\beta)|^p d\beta \gg \int_{X^{-1}}^{X^{-1}} X^p d\beta \gg X^{p-1}.
\]

In addition, note that for positive integers \( s \), by considering the underlying Diophantine system, we have that

\[
\int_0^1 |v(\beta)|^{2s} d\beta \sim C_s X^{2s-1}
\]

so the desired result follows from Hölder’s inequality.

We will use the circle method to prove the main result. To that end, let

\[
\mathcal{M}(q, a) = \{\alpha \in [0, 1] : |q\alpha - a| \leq PX^{-1}\}
\]

with \( P = X^\nu \) for \( \nu > 0 \) sufficiently small, and let

\[
\mathcal{M} = \bigcup_{q \leq P} \bigcup_{a=1}^q \mathcal{M}(q, a), \text{ } m = [0, 1] \setminus \mathcal{M}.
\]
For any measurable \( B \subseteq [0,1) \), let
\[
I_f(p; B) := \int_B |M_f(\alpha)|d\alpha.
\]

We shall prove Theorem 1.1 by using the fact that
\[
I_{\tau}(p) = I_{\tau}(p; \mathbb{R}) + I_{\tau}(p; m),
\]
showing that \( I_{\tau}(p; m) = o(X^{p-1}(\log X)^p) \) and showing that \( I_{\tau}(p; \mathbb{R}) \approx X^{p-1}(\log X)^p \).

3. THE MINOR ARCS

Our bound on the minor arcs will depend on the following result, which is nontrivial for \( X^\varepsilon \ll q \ll X^{1-\varepsilon} \).

**Proposition 3.1.** If \( |qa - a| \leq q^{-1} \) for some \((a, q) = 1, q \geq 1\), then
\[
M_{\tau}(\alpha) \ll X \log(2Xq)(q^{-1} + X^{-1/2} + qX^{-1}).
\]

**Proof.** We have that by (2.1) and the trivial bound \(|E(\alpha)| \leq X^{1/2}\)
\[
M_{\tau}(\alpha) = 2T(\alpha) + O(X^{1/2})
\]
so it suffices to show that \( T(\alpha) \ll X \log(2Xq)(q^{-1} + X^{-1/2} + qX^{-1}) \), since we can absorb the \( O(X^{1/2}) \) into the bound since \( X \log(2Xq)(q^{-1} + X^{-1/2} + qX^{-1}) \gg X^{1/2} \log X \).

To this end, note that by the triangle inequality
\[
|T(\alpha)| \leq \sum_{u \leq X^{-1}} \left| \sum_{u < v \leq X/u} e(\alpha uv) \right| \ll \sum_{u \leq X^{1/2}} \min(X/u, \|\alpha u\|^{-1}).
\]

The desired result then follows from Lemma 2.2 in [Va]. \( \square \)

From this, the following result follows.

**Lemma 3.1.** We have that
\[
I_{\tau}(p; m) \ll X^{p-1} - \nu/2 (\log X)^4.
\]

**Proof.** Note that we have that
\[
\int_m |M_{\tau}(\alpha)|^p d\alpha \leq \left( \sup_{\alpha \in m} |M_{\tau}(\alpha)| \right)^p \int_m |M_{\tau}(\alpha)|^2 d\alpha \ll X(\log X)^3 \left( \sup_{\alpha \in m} |M_{\tau}(\alpha)| \right)^{p-2}.
\]

Suppose that \( \alpha \in m \). Then, by Dirichlet’s theorem, we have that there exist \( a, q \) s.t. \((a, q) = 1, q \leq P^{-1}X, |qa - a| \leq P^{-1}X \), so it follows that \( q \geq P \). Then, by Proposition 3.1, we have that \( |M_{\tau}(\alpha)| \ll X^{1-\nu/2} \log X \), and the desired result follows. \( \square \)

Now, we proceed to estimate the major arcs. To that end, we first record the following estimate.

**Proposition 3.2.** For \((a, q) = 1, q \geq 1\), we have
\[
\sum_{n \leq X} \tau(n)e \left( \frac{an}{q} \right) = \frac{X}{q} \left( \log \frac{X}{q^2} + 2\gamma - 1 \right) + O((X^{1/2} + q) \log 2q).
\]
Proof. This is shown in the proof of Lemma 2.5 in [PV]. We shall reproduce its proof below. Note that we have that
\[
\sum_{n \leq X} \tau(n) e \left( \frac{an}{q} \right) = \sum_{u \leq X^{1/2}} \left( \sum_{v \leq X/u} 2 - \sum_{v \leq X^{1/2}} 1 \right) e(auv/q).
\]
For \( q \nmid u \), we have that the inner sums are \( \ll \|au/q\|^{-1} \). The contribution from the remaining terms is then
\[
\frac{X}{q} \left( \log \frac{X}{q^2} + 2\gamma - 1 \right) + O(X^{1/2})
\]
from which the desired result follows. \( \square \)

Now, it follows then from this and partial summation that for \( \alpha \in \mathcal{M}(q,a) \), we have
\[
M_\tau(\alpha) = \frac{1}{q} \left( \log \frac{X}{q^2} + 2\gamma - 1 \right) v(\alpha - a/q) + O(X^{1/2+\nu}(\log X)).
\]
Therefore, we have that (by using the binomial theorem for \( p \in \mathbb{Z}^+ \), and then using Hölder’s inequality to bound the remaining error terms)
\[
|M_\tau(\alpha)|^p = q^{-p}(\log X - 2\log q + 2\gamma - 1)p|v(\alpha - a/q)|^p + O(X^{p-1/2+\nu}(\log X)^p)
\]
so it follows that
\[
(3.4) \quad \int_{\mathbb{R}} |M_\tau(\alpha)|^p d\alpha = \\
\sum_{q \leq P} \sum_{a=1}^{\phi(q)} \int_{-P^{-1}}^{P^{-1}} q^{-p}(\log X - 2\log q + 2\gamma - 1)p|v(\alpha - a/q)|^p d\beta + O(X^{p-3/2+4\nu}(\log X)^p)
\]
where
\[
\mathcal{S}(X, P) := \sum_{q \leq P} \phi(q)q^{-p}(\log X - 2\log q + 2\gamma - 1).p.
\]
It is easy to show that
\[
(3.5) \quad \mathcal{S}(X, P) \asymp (\log X)^p.
\]
Also, note that since \( |v(\beta)| \leq \min(X, \|\beta\|^{-1}) \), we have that
\[
\int_{-P^{-1}}^{P^{-1}} |v(\beta)|^p d\beta \gg \int_{0}^{1/(4X)} X^p d\beta \gg X^{p-1}.
\]
By considering the underlying diophantine equation, it is quite easy to show that for positive integers \( s > 0 \), we have that
\[
\int_{0}^{1} |v(\alpha)|^{2s} d\alpha \sim C_s X^{2s-1}
\]
for some \( C_s > 0 \). It therefore follows that by Hölder’s inequality since \( p > 2 \)
\[
\int_{0}^{1} |v(\beta)|^p d\beta \asymp X^{p-1}.
\]
Theorem 1.1 then follows.
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References


